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SELF-THINNING LIMITS IN TWO AND THREE DIMENSIONS

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Abstract. The principles behind self-thinning laws and stand density management diagrams are examined. Relationships are analyzed based on trajectories of unthinned and thinned stands in a 3-dimensional state space. Limiting self-thinning lines and planes are demonstrated using a dynamic stand growth model for loblolly pine.

Keywords: Forest growth and yield; Reineke; 3/2 law; Relative spacing; Stand density management diagrams; Thinning; Loblolly pine

1 Introduction

Self-thinning "laws" or rules are a popular topic in forestry. They correspond to limiting straight lines when plotting trees per unit area *vs.* certain stand variables in logarithmic coordinates (Burkhart and Tomé 2012, Section 8.2). Although these relationships are based on empirical observation and have no satisfactory theoretical basis, conformance to their predictions has been proposed as a test of biological realism for growth models (Leary 1997, Monserud et al. 2005, Weiskittel et al. 2011, Section 15.2.3). Even models claiming to be based on physiological processes may rely on them for modelling mortality (e. g. Landsberg and Waring 1997). The rules are also behind stand density management diagrams (SDMDs Drew and Flewelling 1979, Jack and Long 1996).

We examine some of the principles involved, using the *LobDyn* growth model (García et al. 2011) for illustration. The following section presents the most common self-thinning rules and shows to what extent the behavior of *LobDyn*, which was developed independently of such assumptions, agrees with them. Section 3 discusses SDMDs, their uses and limitations. Self-thinning rules and SDMDs are interpreted in Section 4 through projections of 3-dimensional trajectories. This view explains how the various rules are related, and the capabilities of SDMDs for projecting growth of thinned and unthinned stands. Limiting 3-dimensional surfaces previously noted by some authors are presented in Section 5. The article ends with a brief summary and conclusions.

2 Self-thinning laws

The best known self-thinning laws or rules are Reineke's, and the 3/2 law. Reineke (1933) graphed the logarithm of the number of trees per unit area, $\log N$, *vs.* the logarithm of the (quadratic) mean dbh $\log D$, postulating a limiting line with a slope of approximately -1.6. The *3/2 self-thinning law* predicts a limit $\log w + 1.5 \log N = \text{constant}$, where w is mean tree biomass or volume (Drew and Flewelling 1977). A third self-thinning relationship uses stand height H in the form $\log H + k \log N = \text{constant}$, with $k = 2$ corresponding to the Hart-Becking or Wilson index (Beekhuis 1966, García 2009, Wilson 1951). The limiting lines are assumed to be approached when stands undergo "substantial and sustained mortality".

To illustrate such behavior, *LobDyn* was used to generate unthinned predictions for ages 2, 4, ..., 80 years, starting with initial densities of 125, 250, 500, 1000, 2000, and 4000 trees/ha at breast height. Site index was 18, and species composition was 100% pine. In addition, two thinning regimes were simulated, one starting with 1600 trees/ha at breast height and thinning half of the surviving trees at age 20 years, and the other starting with 2500 trees/ha and thinning half of the survivors at age 16.

Figure 1 shows the predicted trajectories in Reineke's $\log D - \log N$ plane (all graphs produced with *Gnuplot*, `http://gnuplot.info/`). The limiting slope is somewhat steeper than -1.6. However, with the variability of real data and stands typically much younger than 80 years, the difference would be difficult to appreciate in practice. Moreover, considerable deviations from -1.6 have been reported in the literature (Burkhart and Tomé

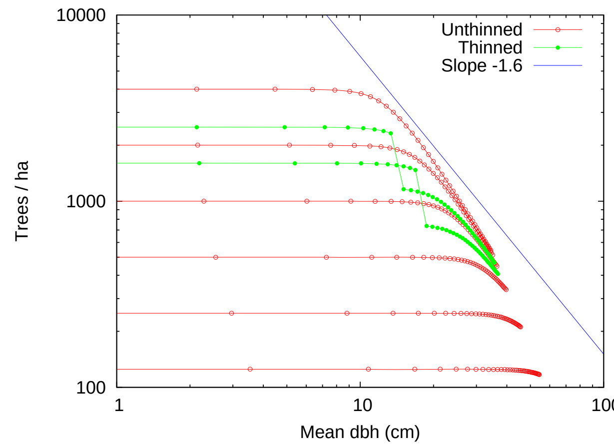

Figure 1: Reineke's graph. Points on predicted trajectories correspond to 2-year age steps.

2012). In *LobDyn* the site index only affects the speed along the trajectories, the trajectories themselves do not change.

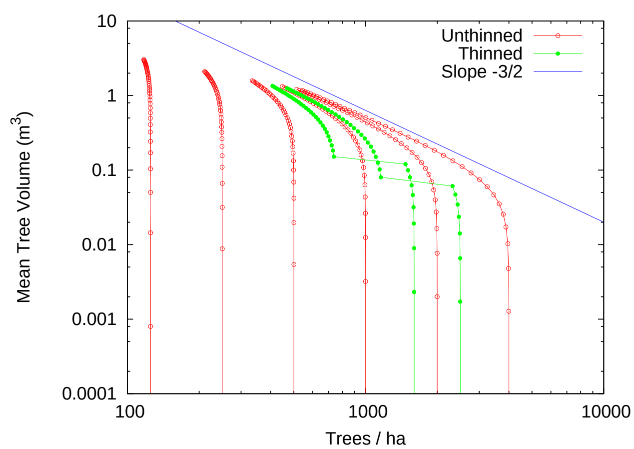

Figure 2: 3/2 self-thinning law. Points on predicted trajectories correspond to 2-year age steps.

The predictions are shown on the usual $\log v$ *vs.* $\log N$ coordinates of the 3/2 law in Figure 2. The variable v is mean stem volume, calculated dividing the volume per hectare by the number of trees. Agreement seems good, especially considering that the older ages are not normally attained.

Trajectories of $\log N$ over the logarithm of top height are shown in Figure 3. There is a limiting line for stands undergoing substantial mortality. The limiting slope, however, is not the -2 implied by the Hart-Becking index, but rather $-(a_2 + 1)/(a_3 - 1) = -3.06$, using the parameters from Section 3.2 of García et al. (2011) (see

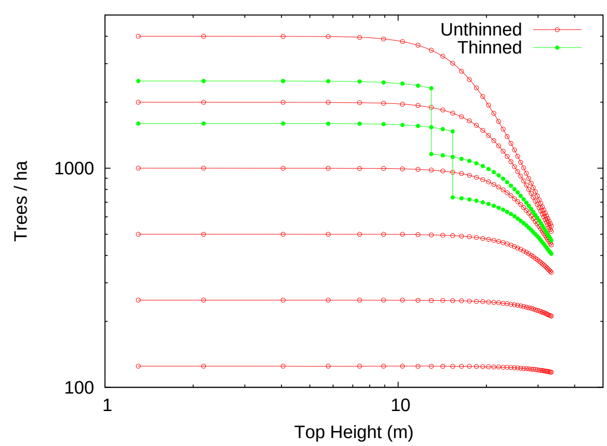

Figure 3: $\log N$ *vs.* $\log H$. Points on predicted trajectories correspond to 2-year age steps.

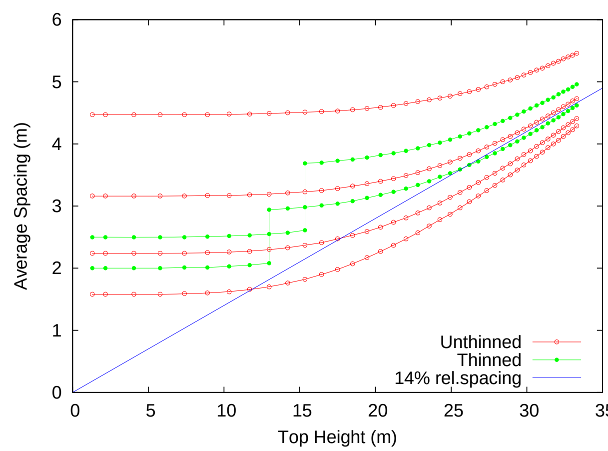

Figure 4: Average (square) spacing over top height, non-logarithmic scales. The slope of the line joining a point to the origin is the relative spacing.

García 2009, Section 4)[1]. Figure 4 indicates that there is no limiting relative spacing (Hart-Becking or Wilson index). This has been found in other species (García 2009). In fact, Reineke, the 3/2 law, and the Hart-Becking index are mutually incompatible (Section 4).

It may be noted tat the practical significance of these self-thinning models is rather limited, at least for managed or planted stands. Most of the interesting stand development occurs far from the limits.

[1]Actually, this is a mathematical limit as $H \to \infty$, but in reality H has an asymptote of 39.40 m. The slopes calculated at this height, for various initial densities N_0, are:

N_0	125	250	500	1000	2000	4000
Slope	-0.56	-1.19	-1.98	-2.57	- 2.87	-2.99

3 Stand density management diagrams

SDMDs try to extend the preceding ideas to stands that are not necessarily self-thinning. Following Reineke or the 3/2 law, stand development is shown in logarithmic axes with number of trees in the abscissa, and mean dbh or mean volume in the ordinate. Sometimes other variables, such as basal area, are used instead of these (Burkhart and Tomé 2012, Drew and Flewelling 1979, Jack and Long 1996).

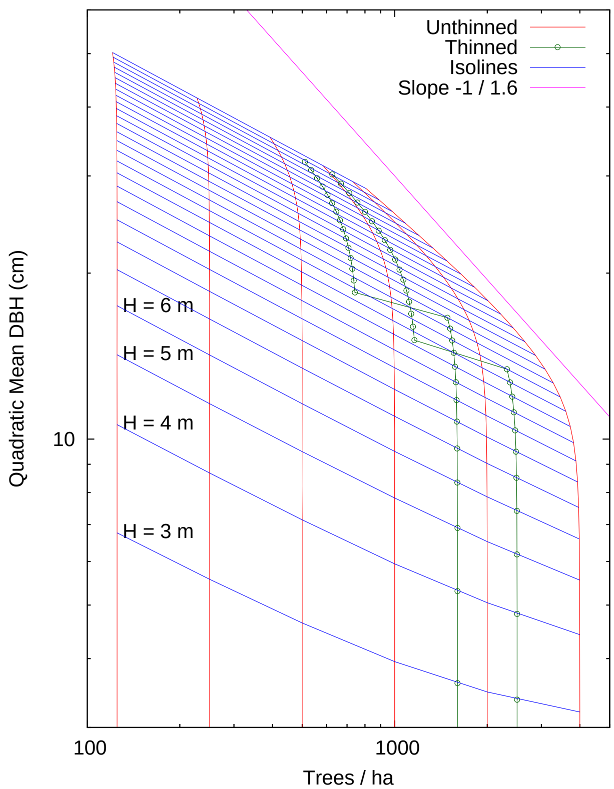

Figure 5: Predicted *LobDyn* trajectories graphed as a Reineke-based stand density management diagram.

The same projections from Section 2 were used, except that they were calculated at equal 1 m top height steps instead of 2-year steps. To reduce clutter the graphs do not show ages older than 60 years. Figure 5 displays the predictions in the form of a D-based SDMD. Figure 6 is the analogous for mean tree volume. Points of equal height are joined by so-called *isolines*. Traditionally, SDMDs de-emphasize the trajectories themselves, representing them as dotted curves or omitting them altogether. Usually, contours are added representing volume or other output variables, that we have not drawn here. Apart from that, the graph for the unthinned stands is similar to a typical SDMD.

SDMDs implicitly assume that the process of removing trees in a thinning follows the isolines. There is no reason why this should be so, and in Figures 5 and 6 the tree size increase estimated by *LobDyn* for typical thinnings is lower than that implied by the SDMD. In addition, until full occupancy has been restored, growth immediately following a thinning is slower than in unthinned stands of the same size and density. The error caused by this logical flaw may or may not be acceptable in the practical application of SDMDs, but it should be kept in mind. García (2003) shows similar results based on simulations with the TASS individual-based growth model (Mitchell 1975).

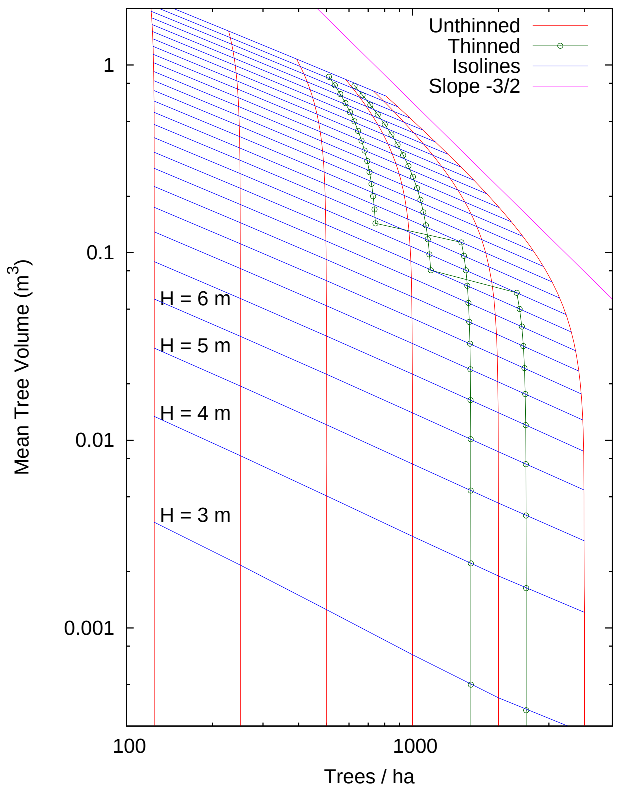

Figure 6: Predicted *LobDyn* trajectories graphed as a 3/2-law-based stand density management diagram.

The logarithmic scale makes it difficult to obtain accurate estimates over much of the range of interest. With modern computer graphics there seems to be little justification for the historical format, and something like Figure 7 might be more useful (García 2003). Or even Figures 8 or 9. Some of the mystique may be lost, though.

4 3-D trajectories and surfaces

Essentially, SDMDs are graphical growth models based on a two-dimensional state space (García 2003). That is, the current values of the variables represented in the two axes of the diagram are assumed to fully deter-

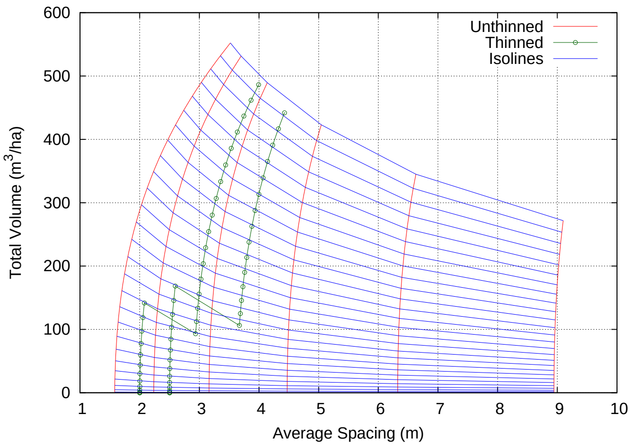

Figure 7: A re-scaled SDMD.

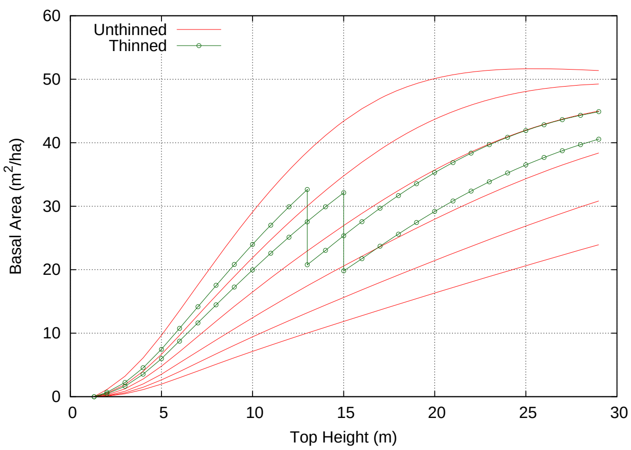

Figure 8: Basal area *vs.* height.

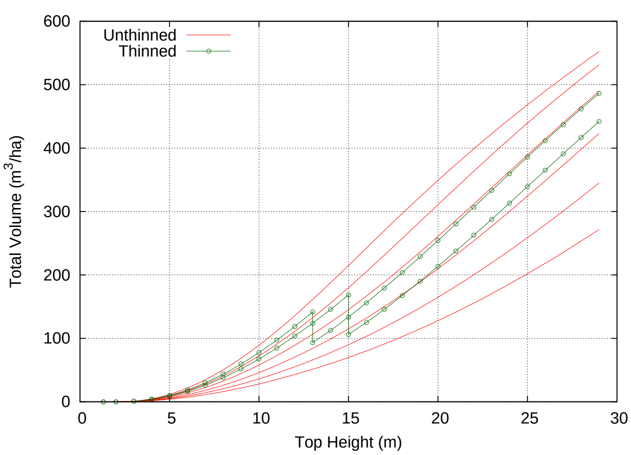

Figure 9: Volume *vs.* height.

mine the evolution of those two variables. That is a good approximation for unthinned stands, but as discussed above, it can fail when stand development is disturbed. Examining behavior in three dimensions can help to understand better the issues involved. As suggested by Abbott (1884, Section 16), things can seem mysterious when looked at from a low-dimensional space (Figure 10).

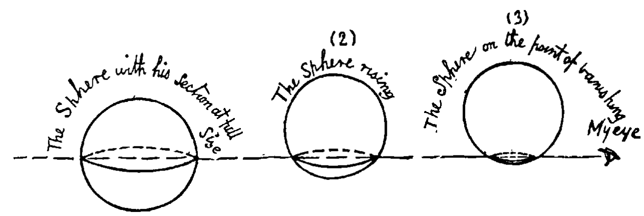

Figure 10: Sphere to Flatlander: "See now, I will rise; and the effect upon your eye will be that my Circle will become smaller and smaller till it dwindles to a point and finally vanishes." (Abbott 1884).

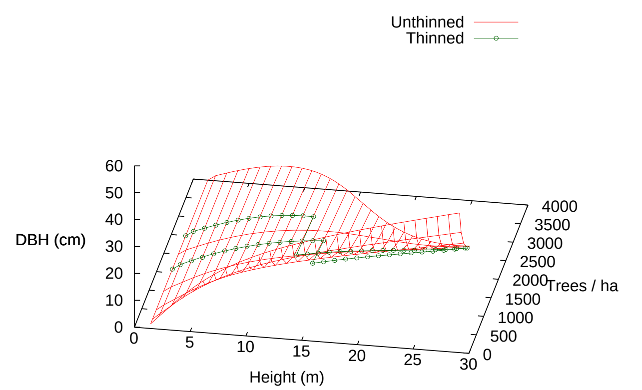

Figure 11: Unthinned trajectories form a 3-D surface. Thinnings drop somewhat below the unthinned surface.

The *LobDyn* predictions from Section 3 are represented in an H–N–D space in Figure 11, and with logarithmic coordinates in Figure 12. Any unthinned stand starting at an initial density N_0 at breast height ($H = 1.3$, $D = 0$) follows a continuous three-dimensional curve in this space. The set of curves for varying values of N_0 form a surface. This observation is completely general (at least for a given site quality), and is not specific to *LobDyn*. Therefore, two state variables are sufficient to describe the dynamics of unthinned stands.

Figures 1 and 5 are projections of Figure 12 into a $\log N - \log D$ plane. Figure 3 is a projection into a $\log H - \log N$ plane. If v is approximately proportional to D^2H, that is, $\log v \approx 2 \log D + \log H + \text{constant}$, then

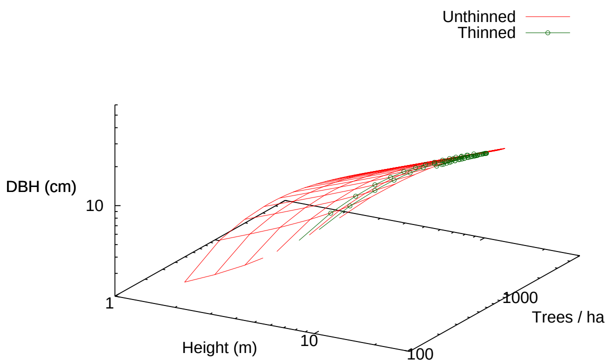

Figure 12: The surface in logarithmic coordinates.

Figures 2 and 6 are projections into another plane. Many different stand density indices might thus be defined. Clearly, any variables that can be expressed as products of powers of D, N and H could be used, for instance, basal area $B \propto D^2N$, or average spacing $S \propto N^{-0.5}$.

Similarly, trajectories, at least those for high densities, can approach a limiting line in the three-dimensional log space. Projections of this line into certain planes generate the various self-thinning lines. Conversely, a two-dimensional self-thinning line defines a perpendicular plane, and two of these planes intersect to generate a three-dimensional limit line. In general, a projection of the 3-D line will not coincide with a third arbitrary 2-D self-thinning line.

Specifically, Reineke's line is

$$\log N + 1.6 \log D = \text{constant} \; , \qquad (1)$$

and the 3/2 law, assuming $v \propto D^2H$, can be written as

$$2 \log D + \log H + \frac{3}{2} \log N = \text{constant} \; . \qquad (2)$$

Eliminating D, the projection into $\log H - \log N$ is found to be

$$\log N + 4 \log H = \text{constant} \; ,$$

which differs from the Hart-Becking-Wilson line

$$\log N + 2 \log H = \text{constant} \qquad (3)$$

(García 1993, 2009). It is possible to obtain compatible self-thinning lines by changing somewhat the various coefficients, and/or the exponents of the approximation $v \propto D^2H$.

Thinning causes stands to drop below the unthinned surface. Two variables are therefore not longer sufficient for describing the dynamics of managed stands. They might be acceptable as a rough approximation, however.

5 The self-thinning plane

Several authors have noted the existence of a 3-D surface, and/or of a limiting "self-thinning plane" in 3-D space (Briegleb 1952, Decourt 1974, García 1988, 1993, O'Hara and Oliver 1988)[2]. The explanation of Decourt (1974) assumes stands with different thinning regimes, all starting from the same initial density. O'Hara and Oliver (1988) used age instead of H, see also Oliver and Larson (1996, Figure 15.1).

By rotating Figure 12, it is found that after canopy closure the surface is close to a plane (Figure 13). This is also suggested by he nearly straight and parallel isolines in Figures 5 and 6, although those do not rule-out curvature in the H-direction. "Self-thinning plane" is perhaps not quite accurate, because the approximation is good also for stands not undergoing self-thinning.

An equation for the plane can be obtained by linear regression of one of the log-transformed variables over the other two (excluding young stands for which the approximation does not apply). Or a little more elegantly, by finding the direction that minimizes the sum of squared deviations. That direction is given by the covariance matrix eigenvector with the smallest eigenvalue, or equivalently, by the less significant principal component (García 1993). Calculating in R (R Development Core Team 2009) using the predicted values with $H > 6$ m,

```
> x <- log(LobDyn[LobDyn$H > 6, c('N', 'H', 'D')])
> (y <- eigen(cov(x)))
$values
[1] 1.1392336301 0.2322456491 0.0002257582

$vectors
           [,1]       [,2]       [,3]
[1,]  0.9193056 -0.3069757 -0.2462585
[2,] -0.1541529 -0.8566269  0.4923690
[3,] -0.3620969 -0.4146761 -0.8348230

> summary(as.matrix(x) %*% y$vectors[,3])
       V1
 Min.   :-2.818
 1st Qu.:-2.809
 Median :-2.800
 Mean   :-2.797
 3rd Qu.:-2.788
 Max.   :-2.742
> c(y$vectors[,3], -2.797) / y$vectors[1, 3]
[1]  1.000000 -1.999399  3.390028 11.357986
```

Scaled so as to have a unit coefficient for $\log N$, the

[2] This is different from Bi (2001), where the third coordinate is site index.

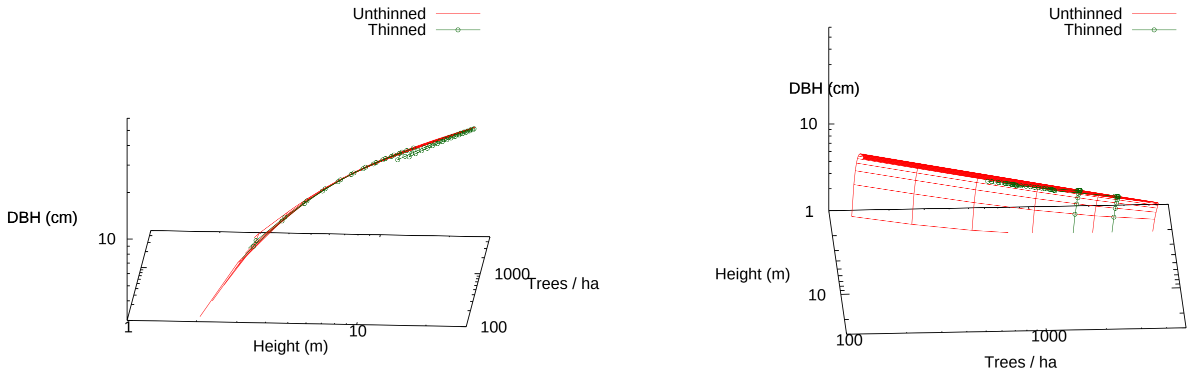

Figure 13: Plane approximation.

equation of the plane is

$$\log N - 2.00 \log H + 3.39 \log D = 11.36 \ . \qquad (4)$$

This is similar to the equation $\log N - 2.29 \log H + 3.28 \log D =$ constant reported for radiata pine permanent sample plots by García (1993). A principal components fit to Table 2 of Briegleb (1952) gives $\log N - 1.50 \log H + 2.78 \log D =$ constant.

Similarly, the values for $N_0 = 4000$ and $H > 15$ give two eigenvectors with small eigenvalues, corresponding to two planes that intersect to define a three-dimensional self-thinning line. The projections on the standard planes are found to be

$$\log N + 1.91 \log D = \text{constant} \qquad (5)$$

$$2 \log D + \log H + 1.48 \log N = \text{constant} \qquad (6)$$

$$\log N + 2.35 \log H = \text{constant} \ , \qquad (7)$$

corresponding to equations (1), (2), and (3).

6 Summary and conclusions

Predictions from *LobDyn* are found to conform reasonably well to traditional self-thinning "laws", even though these are not built into the model. Rather than fundamental biological principles, the rules should be seen as empirical limits that may be acceptable approximations under certain circumstances. In particular, taking an amount of (largely dead) xylem acumulated on the stems, represented by D or v, as a driver or explanatory variable may be seen as questionable from a physiological point of view (García 2009, García et al. 2011). The three conventional laws are not mutually compatible, and can be interpreted as plane projections of a three-dimensional line in logarithmic coordinates.

Expected unthinned trajectories are necessarily restricted to a surface in three dimensions. This fact can be useful for understanding the functioning and limitations of stand density management diagrams (SDMDs). Thinning causes deviations away from the surface, that are not properly handled by SDMDs. If low accuracy is sufficient, however, these deviations might be considered as relatively small compared to the full range of growing conditions. Almost anything plotted on logarithmic coordinates seems to tend to a straight line. On the other hand, the logarithmic scale compression can obscure relevant stand behavior.

In common with previous observations in the literature, *LobDyn* trajectories approach a three-dimensional "self-thinning plane".

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