ISSN 1946-7664. MCFNS 2021
Vol. 13, Issue 1, pp. 36–43
Mathematical and Computational Forestry & Natural-Resource Sciences
AVAILABLE ONLINE AT HTTP://MCFNS.COM
Submitted: Mar. 5, 2021
Accepted: Mar. 30, 2021
Published: Mar. 30, 2021
Last Correction: Apr. 5, 2021

UTADA: UNIFIED THEORY OF THE ALGEBRAIC DIFFERENCES APPROACHES—DERIVATION OF DYNAMIC SITE EQUATIONS FROM DIRECT YIELD-SITE RELATIONSHIPS

Chris J Cieszewski
University of Georgia, Athens, GA 30602 USA

Abstract. Dynamic-equation-based self-referencing models of the form $Y = f(t, t_0, y_0)$ describe changes in Y as a function of a longitudinal variable t and an unobservable cross-sectional variable X, which is implicitly represented by a known snapshot observation of Y, y_0, at an arbitrary value of t, t_0. The unobservable variable X denotes the environment potential, or site, which cannot be directly measured or precisely defined due to its extreme complexity and variability. While elusive and difficult in handling, X is the most critical variable of the site equations due to its disproportionate impact on the modeled dynamics. All traditional approaches to such modeling are predominantly based on a detailed analysis of primarily longitudinal relationships $Y = u(t)$, which subsequently, to be helpful in practice, are modified into the self-referencing forms, thus incidentally accounting for the site impacts. All the former approaches devote little to no effort to explicitly model the cross-sectional relationships governed by the unobservable variable X.

I hereby present a proof of a concept for a novel approach to derivation of the dynamic-equation-based self-referencing models that unifies the modeling efforts of defining the yield and site relationships equally, by focusing primarily on direct mathematical formulations describing the theory of the yield-site relationships. This approach considers the variable t only in the secondary analysis, adding it to the framework through modifications of the final model parameters. Despite the somewhat elusive nature of exploring the unobservable variable properties of the site, the new approach appears to be highly empowering by analyzing simple and direct yet more robust relationships between Y and X as opposed to those between Y and t. The self-referencing dynamic site equations derived through this approach have all the desirable properties of site models, such as the base-age-invariance, path-invariance, and a high degree of flexibility with complex polymorphism and variable asymptotes.

Keywords: Site models; site index modeling; GADA models; self-referencing functions; base-age-invariance; path-invariance.

1 Background

A scientist analyzing data of multiple development series on various sites wants to summarize the trends using a single practically useful mathematical model. The shape of curves representing various development conditions may vary due to differences between various environments, including such varying components as nutrient and moisture availability, growth inhibitors and competition, individual microclimates, etc. If changes between individual series are consistent and continuous, a general model, rather than multiple location-specific models, can illustrate the trends across all sites. However, because the development conditions are composed of many variables that are not measurable within practical limitations, it is expedient to use an observed yield at a known time to measure the individual growth potential. This leads to the development of the self-referencing models (Northway 1985). In such models, the Y variable is a function of both variables X and t, but in practice, they define the value of Y at a time t as a function of a known sample yield $Y = y_0$, at an arbitrary time $t = t_0$ (i.e., $Y = f(t, t_0, y_0)$).

The dynamic self-referencing models have the potential for applications in many fields where the unobservable independent variable X may refer to a host, or development conditions, nutrients and water availability, site quality, or productivity, or, plainly, the environment. In general, the modeled dependent variable may be the state or yield of a phenomenon affected by the two independent variables, of which one is unobservable and potentially multivariate or multi-dimensional. Such models play an essential role in forest management, where they have been in common use since around the 1930s (e.g., Schumacher 1939) and are used for modeling site-dependent development of nearly all stand and tree characteristics considered in forest management.

The model forms and methods of self-referencing site equation developments have been changing over time. Bailey and Clutter (1974) formalized the Schumacher's original approach as the Algebraic Difference Approach (ADA). ADA is a parameter-based approach relying on re-parameterizations of age-based functions into the self-referencing forms, similar to the traditional initial-condition equations. Practically all the dynamic site equations derived with ADA were either anamorphic or had fixed asymptotes. The first complex polymorphic dynamic site equation with variable asymptotes was proposed in the 1980s (Cieszewski 1987 and Cieszewski and Bella 1989). Subsequently, Cieszewski and Bailey (2000) formalized the method of Cieszewski (1994) as the Generalized Algebraic Difference Approach (GADA), which is based on an initial assumption of expanding the yield-age models to explicit static yield-age-site models (i.e., $Y = f(t, X)$) before the derivation of the dynamic site equation (i.e., $Y = f(t, t_0, y_0)$). The development of GADA enhanced capabilities to model more complex site-dependent dynamics was a significant breakthrough in the evolution of the self-referencing dynamic site equations, and it led to derivations of many such new models with complex polymorphism and variable asymptotes.

The tendency of modeling height-age relationships and adapting them to site variation, as opposed to, for example, modeling height-site relationships and adapting them to age responses, has been deeply rooted in the historically conditioned modeling culture associated with these models. Thus, for example, the ADA approach has been used strictly as age-function oriented and it didn't lend itself much to thinking about the explicit site effects modeling other than in terms of one of the yield-age model parameters. Similarly, the traditional static site equations (i.e., models with fixed base-age site index as site parameter: $Y = f(t, S)$), which go back earlier than the dynamic equations, also entirely rely on age-dependent relationships that are modified by adding a fixed-base-age site index to height-age models. The same can be said about the oldest efforts of representing the height growth series in tables, which always focused only on the height-age timeseries rather than, for example, on isolines of the same age heights across different sites. Even GADA approach, which makes a step in the right direction by encouraging the formulation of an explicit $Y = f(t, X)$, before the development of dynamic site equations, and treats the site as a modeling variable of similar importance to that of age, is usually executed with the primary attention given to the yield-age relationship and its parameters, using the site X as a modifier of that relationship. In general, the primary efforts in developing the dynamic site equations have been typically limited to looking for more suitable yield-age equations with expectations that better yield-age equations would produce better site equations (Clutter et al 1983). Finally, even those who found previously published site equations fit their data well are likely to attribute the model suitability for their data to the model yield-age description, rather than the yield-site relationship characteristics.

The fact that many researchers view the self-referencing functions as such that can be derived almost exclusively from known age-dependent equations is ungrounded because there are many ways to derive any model starting with different base-models. Some authors have even ignored the actual source of applied self-referencing dynamic equations they used, misnaming them according to deemed bases of their potential underlying yield-age relationships. Yet, many yield-age models will fit most of the single series data similarly well, while modeling changes across sites — especially using only implicit site definitions — is far more challenging and may result in dramatically different fitting success for various definitions of yield-site relationships regardless of the yield-age base model forms. It seems that not seeing the site modeling aspect from behind the height-age problem framing is the main impediment hindering the progress in many site modeling studies, according to which many site modeling problems have been lingering for almost a century. The emergence of GADA has mitigated the yield-age modeling biases by forcing the modelers to consider the site modeling explicitly at the stage of the yield-age-site base model formulation. However, only the presented here methodology is unequivocally forcing the modelers to consider the yield-site relationship as the primary effort in deriving the self-referencing dynamic site equations.

The reasons for avoiding modeling directly the yield-site relationships in the past likely include the following:

1. Age is a well understood and easily measured quantity while site quality is an unobservable variable that we don't fully understand and cannot explicitly measure, which discourages efforts in its direct modeling;

2. There are many readily available age-dependent models explicitly dealing with age, while there are no well-established models of site responses other than those implicit in parts of yield-age-site rela-

tionships, which are usually taken for granted and almost never individually considered;

3. The age-dependent relationships have been modeled for centuries while site-dependent relationships have not been explicitly modeled for nearly as long and few have asked a question such as: "What could be an explicit yield-site relationship?"

On the surface, to many practitioners, the last question is not very interesting without considering age-dependent changes. After all, what forest managers want to know are the values of these changes for any given timeframe. The site quality is just a nuisance that makes predictions more complex and needs to be accounted for. How can we talk about site responses if we don't know what exactly the site is and all that a site does is modifying the age-dependent processes?

Yet, modelers have the luxury of doing things that may seem esoteric in the context of operations but allow theoretical structuring of specific problems in ways that may reveal newly possible results. Accordingly, it seems reasonable to ask if (and if *yes*, then how?) we can model the site quality responses directly without treating site quality input as a fifth wheel on a yield-age vehicle. There are good reasons to believe that this should be the direction of our efforts. One reason is that modeling the site responses is the biggest challenge. The second reason is that poorly modeled site responses have a statistical impact far more significant than poorly modeled age responses. Practitioners with a substantial experience in developing site-dependent yield-age models can attest to this, as typically the errors in fitting all sites together are in orders of magnitude greater than those from fitting individual site-specific yield-age curves. The costs of adding site responses to yield-age models are typically so great that we are unlikely to do worse if we reverse the process and add the age responses to yield-site models.

2 Objectives

The objective of this study is to demonstrate a new and unorthodox unified approach to the direct derivation of self-referencing dynamic yield-site relationships formulating its primary framework for derivations of dynamic site equations. The new approach unifies the efforts towards modeling yield and site dynamics with respect to each other. In this sense, the method is contrary to traditional practices. It consists, most of all, of modeling the yield-site responses directly instead of giving priority to modeling the yield-age responses. The approach adds the age-dependent responses as merely parameter-modifying functions in the final dynamic yield-site equations (instead of secondarily adding the site responses to the primary yield-age models).

3 Methods

3.1 The Theory

The most critical processes captured by algebraic differences equations are the unified relationships between the modeled phenomena (i.e., yield) and its environment, or host (i.e., site). Accordingly, the yield-site relationship should be treated as the primary objective of the self-referencing dynamic equation derivations, while the yield-age relationship should be the secondary objective.

The above assertion is contrary to the past modeling practices in derivations of the self-referencing equations that were based primarily on modeling the relationships between yield and age and just modified to accommodate site. Both site and yield are equally essential and should be modeled within a unified framework of the functional dependence of yield on site. Moreover, this relationship is likely to be mathematically simpler and more stable than, for example, the changes of yield over time, while the later responses can be added to the unified framework at a latter, secondary rather than primary, stage of modeling.

3.2 Model Bases

Let the dependent variable, typically yield or height, be Y, and the site quality be X. Any reasonable measure of site quality (that we may want to model) must be correlated with the yield of the modeled phenomena (i.e., $Y\tilde{}X$), which, by definition, depends on site quality. It is also reasonable to assume that for certain choices of Y, the ratio $Y/X = Constant$ (see Fig. 1), similar as it is for, let's say, height at base age and site index, which is for all sites: $H50/S50 = 1$. Accordingly, similar to the site index example, for any given site we can assume the following expectations:

1. for certain selections of Y, for example, at specific ages, we could observe $Y/X = 1$, which would be equivalent to $X/X = 1$;

2. for other selections of Y, for example, younger ages, we could observe $Y/X < 1$, which could be equivalent to $X/(X + a) < 1$; and

3. for yet another selection of Y, for example, older ages, we could observe $Y/X > 1$, which could be equivalent to $(X + b)/X > 1$.

There are many ways in which we could model these kinds of relations, with one of the most straightforward

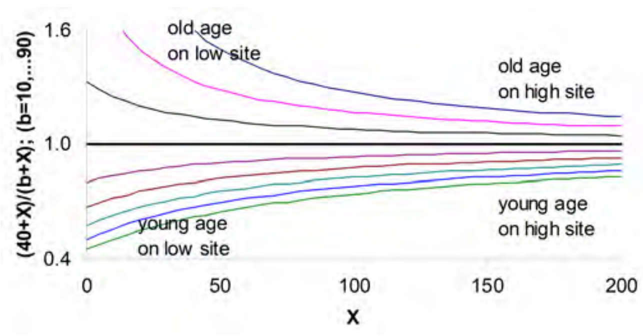

Figure 1: A relationship between a relative yield Y/X and a simple site-parameter model deviating from an assumed default constant $(a+X)/(b+X)$, where a is a fixed parameter and b is assumed to be the varying longitudinal parameter.

generalizations, according to the above assumptions, being $\frac{Y}{X} = \frac{X+a}{X+b}$, or more generally, replacing with a linear function of site either the right-hand-side numerator $(c \cdot X + a)$ or denominator $(c \cdot X + b)$ of this relationship, which could result in the following simple example:

$$\frac{Y}{X} = \frac{X+a}{c \cdot X + b} \quad , \tag{1}$$

where:

- a, b, and c, are the model parameters modifying the yield over site relationship, with at least one or more of them being a time-related varying-parameter (e.g., varying with different selections of t); and
- X is the site quality, such that$X = 1/2 \cdot \left(K \pm \sqrt{K^2 + 4b \cdot Y}\right)$, where$K = c \cdot Y - a$. However, since for positive values of b and Y, which is a reasonable assumption to make, only the positive root is desirable for consideration, I consider hereafter only$X = 1/2 \cdot \left(K + \sqrt{K^2 + 4b \cdot Y}\right)$.

Assuming for simplicity that only parameter b varies in Eq. (1) (see Fig. 1), we can define for deriving dynamic equation a specific solution for site, designating it as the reference point, as follows:

$$X_0 = 1/2 \cdot \left(K_0 + \sqrt{K_0^2 + 4b_0 \cdot y_0}\right) , \tag{2}$$

where:

- $K_0 = c \cdot y_0 - a$;
- y_0 is a known value of Y at an arbitrary age $t = t_0$, and it is used as an implicit X site quality measure; and
- b_0 is the model varying-parameter b marked with the subscript as belonging to the specific solution for site quality X_0, which is necessary for the usefulness of this solution in further development of the dynamic self-referencing site equation.

The dynamic self-referencing equation defining a yield-site relationship (Fig. 2) with an implicit definition of site quality is then derived substituting X in Eq. (1) with the specific solution X_0 in Eq. 2, which produces the following self-referencing dynamic equation of yield as a function of site measured implicitly by a known yield observation:

$$Y = y_0 \frac{c \cdot X_0 + b_0}{c \cdot X_0 + b} \quad , \tag{3}$$

where:

- X_0 is defined by Eq. (2) with varying parameters b and b_0 to be substituted by a function of a longitudinal variable, such as for example, a function of time;
- the subscript "$_0$" on the model parameter b_0 is maintained to keep track of the complete definition of the specific solution for site quality, without which Model (3) would not be practically usable in operational implementations; and
- all other symbols are as previously defined.

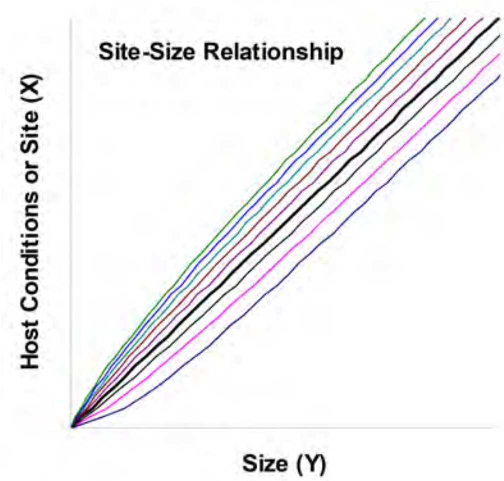

Figure 2: A hypothetical relationship between site or host conditions and yield.

Model (3) represents the self-referencing dynamic equation based on the simple assumptions about yield-site relationships defined by Eq. (1). It is ready for a

Table 1: Examples of basic functions of a single variable that can be used for substituting the parameters b, b_0, c and c_0 in the size-site models (3)–(5) to incorporate the additional longitudinal variable t; the functions in the table have been adapted to the inverse requirement for the parameter substitution.

	Function Name	Function General Definition	Definition of b Definition of c	Definition of b_0 Definition of c_0
1.	Linear	$F(x) = p_1 + p_2 \cdot x$	$b = b_1/(b_2 + t)$ $c = c_1/(c_2 + t)$	$b_0 = b_1/(b_2 + t_0)$ $c_0 = c_1/(c_2 + t_0)$
2.	Quadratic	$F(x) = p_1 + p_2 \cdot x + p_3 \cdot x^2$	$b = b_1/(b_2 + b_3 \cdot t + t^2)$ $c = c_1/(c_2 + c_3 \cdot t + t^2)$	$b_0 = b_1/(b_2 + b_3 \cdot t_0 + t_0^2)$ $c_0 = c_1/(c_2 + c_3 \cdot t_0 + t_0^2)$
3.	Power	$F(x) = p_1 \cdot x^{p_2}$	$b = b_1/t^j$ $c = c_1/t^w$	$b_0 = b_1/t_0^j$ $c_0 = c_1/t_0^w$
4.	Exponential	$F(x) = p_1 \cdot p_2^x$	$b = b_1/b_2^t$ $c = c_1/c_2^t$	$b_0 = b_1/b_2^{t_0}$ $c_0 = c_1/c_2^{t_0}$
5.	Logarithmic	$F(x) = p_1 \cdot \ln(t + 1)$	$b = b_1/\ln(t + 1)$ $c = c_1/\ln(t + 1)$	$b_0 = b_1/\ln(t_0 + 1)$ $c_0 = c_1/\ln(t_0 + 1)$

straightforward implementation into any specific modeling situation using other variables, such as age, diameter, or basal area. The other variables' implementation into the model consists merely of defining the varying-parameter b (and b_0) as appropriate basic functions of age, diameter, basal area, or any other longitudinal measure of interest.

4 Results

Model (3) is derived from direct modeling of unified yield-site relationships with relatively simple and easily adaptable assumptions. One can easily modify this model to incorporate additional variables by replacing the varying-parameters with site-independent parameters and any of their arbitrary functions. Table 1 contains examples of potential definitions of b, b_0, c, and c_0, based on the five basic mathematical functions listed there.

For the presentation clarity, let us assume that in Model (3) $c = 1$. A simple example of a suitable substitution for the varying parameter b and b_0, could be a function increasing nonlinearly with time, which would be a reasonably expected behavior. Given one of the most straightforward functions, let us say, t^2, and the fact that the parameters with the subscript "$_0$" correspond to variables with the same subscript, the substitutions would be: $b \to b_1/t^2$ and $b_0 \to b_1/t_0^2$, which would lead to derivation of the following dynamic equation:

$$Y_{(t,t_0,y_0)} = y_0 \frac{a - y_0 - 2\frac{b_1}{{t_0}^2} - \sqrt{4\frac{y_0 b_1}{{t_0}^2} + (a - y_0)^2}}{a - y_0 - 2\frac{b_1}{t^2} - \sqrt{4\frac{y_0 b_1}{{t_0}^2} + (a - y_0)^2}} \quad ,$$

which in a more compact notation is:

$$Y_{(t,t_0,y_0)} = y_0 \frac{\left(b_1 + X_0 \cdot t_0^2\right) \cdot t^2}{\left(b_1 + X_0 \cdot t^2\right) \cdot t_0^2} \quad , \tag{4}$$

where:

- $X_0 = K_0 + \sqrt{K_0^2 + 4b_1 \cdot y_0/t_0^2}$ and $K_0 = y_0 - a$;
- a, and b_1 are the model's estimable global parameters;
- y_0 is the model's estimable local parameter, and t_0 is a constant (or vice versa);
- t is the independent variable and Y is the dependent variable; and
- t_0 and y_0 are the model's reference points, such that $y_0 = Y$ if $t = t_0$, or $Y(t_0) = y_0$.

Model (3) can define many other models. To derive different models from Eq. (3), one needs to substitute the parameters: b and b_0, with any other arbitrarily suitable basic functions of age, or other variables, directed by expectations of the model reasonable responses with respect to the simulated processes. Various additional substitutions are also possible using any of the basic age functions in Table 1, or other similar functions. Notably, the new models do not have to be designed only for growth but can also be formulated to model survival and other processes.

Finally, the functionality of Model (3) can be easily expanded by modifying assumptions about the varying parameters. A simple example of that could be defining parameter c as another varying parameter, which would result in a different more general model than Eq. (3). Symbolically:

$$Y = y_0 \frac{c_0 \cdot X_0 + b_0}{c \cdot X_0 + b} \quad , \tag{5}$$

where:

- *c, c0, b*, and b_0, are the model varying-parameters, which need to be substituted by appropriate functions of time as in the above example of derivation of Model (4) and examples of the basic functions in Table 1; and
- X_0 is defined by Eq. (2) with $K_0 = c_0 \cdot y_0 - a$;
- the subscript "$_0$" on the model parameter b_0 and c_0 is maintained to keep track of the complete definition of the specific solution for site quality, without which Model (5) would not be practically usable in operational implementations; and
- all other symbols are as previously defined.

Model (5) and (3) represent the simple primary direct yield-site modeling assumptions described by Eq. (1) with different assumptions about the time-dependent varying parameters. The implementations into operational uses are similar for both models. Derivation of Model (4) illustrates example of such implementation, which is based on substituting the selected varying parameters (i.e., b,b_0, c, and c_0) with arbitrary, yet appropriate, functions of other variables, such as age, diameter, or basal area, or any other longitudinal variable of interest (e.g., Tab. 1).

5 Discussion

The presented here study constitutes a proof of concept for a novel approach to derivation of advanced self-referencing dynamic equations that are essential in forest management, where they have multiple uses. In growth and yield modeling, they are used to model many parameters relating to forest inventory, including diameters, heights, basal area, volume, and mortality. In other fields, they can also have applications such as, for example, in the medical sciences, one can use them to predict expected future individual height based on the infant height measurements.

Forest managers have been trying for centuries to develop the best possible methods of using data describing a population's state to predict or estimate this population's future and past conditions. Such knowledge is necessary for efficient forest management and guidance in silvicultural practices applied to vast areas of forest populations. The need of, and the interest in, self-referencing functions is more than a century old. It reaches the oldest times when forest managers first wanted to recognize the site influence on forest populations' growth and development in either tables or graphs containing information on time-series developments on multiple sites. Such models in the form of tables and graphs were used with interpolation between different series based on actual observation of growth in the field. The mathematical models in this category have drastically evolved since. The directions of changes followed different paths with a general regularity of developing from simple anamorphic, through simple polymorphic, to complex polymorphic, while also transitioning from static models based on mathematically intractable fixed base-age site indices to dynamic equations based on direct uses of any age-height combinations resulting in base-age-invariant and path-invariant predictions defining unequivocally consistent trajectories.

The static models used to lead the evolutional changes in reaching greater model flexibility. They started at the beginning of century, about 40 years before the dynamic equations. The first static polymorphic site equations with variable asymptotes originated in the 1970s. The first dynamic equation with polymorphism and variable asymptotes began a decade later (Cieszewski 1988, Cieszewski and Bella 1989). Although the dynamic models lagged behind the static models with their origin and subsequent advancements due to their much more demanding mathematical complexity, the dynamic equations eventually superseded the static models in the number of forms, complexity, flexibility, and the number of alternatives for their development. With the emergence of GADA models and subsequently the presented here new approach to the direct derivation of yield-site dynamic equations — easily modifiable to implement a variety of modifying functions of age — the static site equations are becoming obsolete.

The approach proposed here breaks the century-long tradition of modeling the site-dependent changes through primarily modeling age-dependent relationships and merely modifying them to accommodate the site impacts. Instead, the new approach focuses directly on the most important and challenging question of the yield-site relationships. Surprisingly, this seemingly counter-intuitive approach is founded on extreme simplicity, such as modeling a set of lines deviating from a central horizontal line of a constant relationship (see Fig. 2). Yet, this approach leads to the derivation of extremely powerful self-referencing dynamic equations with unprecedented flexibility.

6 Conclusions

Deriving self-referencing models, changing priority from primary yield-age modeling (subsequently modified to implement site effects) to primary yield-site modeling (subsequently modified to accommodate time changes) unifies the yield dynamics with the site effects resulting in more straightforward yet more powerful modeling possibilities. The unified theory of algebraic difference approaches differs from all previously known methods. It focuses primarily on modeling the yield-site relationships directly instead of focusing primarily on the age-dependent functions. The deriving of self-referencing advanced dynamic site equations through primarily modeling the yield-site unified framework relationship is simpler and more potent than all former possibilities provided by other more traditional age-based approaches.

Acknowledgements

I am grateful for the constructive technical review comments from Dr. Trenton Schirmer of the Department of Mathematics at the University of Georgia and Dr. Mike Strub, retired from Weyerhaeuser Company, for the thorough English corrections from Mr. Marc de Brisay, and for overseeing the peer-review process of this article by Dr. Bogdan Strimbu. All the comments and corrections were very helpful in improving this manuscript. Thanks are also due to an anonymous reviewer who, despite a lack of familiarity with this paper's subject, made sincere efforts to provide numerous comments on clarity and presentation of an earlier version of this manuscript.

Dedication

I am dedicating this article to the memory of Jeremy L. Clutter, one of the first advocates of using dynamic self-referencing equations in forestry. He was one of the precursors of Mathematical and Computation Forestry, and his contributions to Quantitative Forest Management cannot be overrated. I dare to believe that he would appreciate the emergence of GADA and, even more so, the simplicity-in-potential of the presented here UTADA. Sadly, he left us much too early. In 1997, I was honored to join his Warnell team, still intact then, and collaborate with its members before they retired. These included (alphabetically): B.R. Bailey, B.E. Borders, G.H. Brister, J.C. Fortson, L.V. Pienaar, B.D. Shiver, and K.D. Ware. Ever since, I have considered myself fortunate to have met and known them. It was likely the most numerous Forest Biometrics faculty group that had ever existed at any university.

Autobiographical Note

I have been interested in the self-referencing site equations for over 40 years since I first started working on harvest scheduling problems around 1981. I have always found these models intriguing, powerful, and highly practical. Although unintentionally, Dr. Hamish Kimmins of the University of British Columbia has sparked my greatest fascination with these models in 1985. He convincingly argued in a field lecture that it was a wise and pragmatic philosophy to let a plant itself measure and tell the scientist what the site quality was. Ironically, I think he was in this context somehow criticizing the site index models. Still, for me, the take-home message was that the self-referencing models were the ultimate expression of this kind of philosophy, which has since resonated even more with me. They are consistent with the scientist's humility, asking the modeled organism growth to be the integrator of incomprehensibly rich site quality complexity. I have since pursued this subject with enthusiasm, invented new advanced models, and developed a new methodology that I used for derivation of even more new advanced models.

Given my long and extensive experience with the site-dependent modeling, I decided to write this article in an opinionated review style. I give only a few citations of sources that I thought would be necessary for this presentation. I felt that this way, the derivations, arguments, and explanations, would be less obstructed and cluttered and easier to follow. All the accounts of what has been done in this area come from my familiarity with the subject. All those who have more than 40 years of experience in this area, or produced more in this field than I did, and contributed more advancements in this area than I did through GADA invention and other related developments, hopefully, will forgive me this presentation style. Other readers, with possibly lesser experience, I beg to have patience with this presentation and to consider what I believe is true and set forth as helpful background for the presented material. The proposed method of dynamic self-referencing site model derivation is the best idea I have ever discovered regarding this subject.

References

Bailey, R.L. & J.L. Clutter. 1974. Base-age invariant polymorphic site curves. For. Sci. 20(2): 155–159.

Cieszewski, C.J. 1988. New Polymorphic Height Growth and Site Index Model for Lodgepole Pine in Alberta. M.F. Major paper. University of British Columbia, Vancouver, B.C. 28p.

Cieszewski, C.J. 1994. Development of a variable density height-growth-model through defining multidimensional height growth spaces. Ph.D. thesis. University of Alberta, Edmonton, AB. 72p.

Cieszewski, C.J. & R.L. Bailey. 2000. Generalized Algebraic Difference Approach: A New Methodology for Derivation of Biologically Based Dynamic Site Equations. Forest Science 46:116–126.

Cieszewski, C.J., & I.E. Bella. 1989. Polymorphic Height Growth and Site Index Curves for Lodgepole Pine in Alberta. Can. J. For. Res. 19: 1151–1160.

Clutter, J.L., J.C. Fortson, L.V. Pienaar, G.H. Brister, & R.L. Bailey. 1983. Timber management: A quantitative approach. John Wiley&Sons Inc. New York. 333p.

Northway, S.M. 1985. Fitting Site-index Equations and Other Self-Referencing Functions. For. Sci. 31: 233–235.

Schumacher, F.X. 1939. A New Growth Curve and its Application to Timber Yield Studies. J. For. 37: 819–820.